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Star transformations and their genealogical varieties in symbolic dynamics of four letters

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Abstract. The composition rules of period-tripling and period-doubling in symbolic dynamics of four letters are proposed. The methods to generate various singly and doubly superstable kneading sequences from triply superstable kneading sequences are given. These composition rules and methods will present all the essential elements in the three-dimensional parameter space, including the *'joint'*, *'bones'*, *'membrane surfaces'* on the *'skeleton'* and many fractal objects constructed by period-doubling and period-tripling bifurcations. It is shown that the generalization from unimodal and bimodal to trimodal maps is not trivial. And the promptly exponential growth in the kinds of star transformations is even seen when the number of critical points (or parameters) of the map increases.

1. Introduction

Many achievements have been made in the study of the symbolic dynamics of one-dimensional unimodal and bimodal maps [1–7]. In the studies of multimodal maps, symbolic dynamics of trimodal maps has received increasing attention [8–10].

In classical and quantum physics there are a number of interesting systems; their dynamical behaviour can approximately be modelled by trimodal and even multimodal maps with different parameters. Suppose that the physical system has a rather more complex potential than a single well, such as a cubic, quartic or other higher power potential (multiwell potential). They may have the property of a multimodal map. However, we cannot see immediately the connection between the real physical system and their corresponding iterative dynamics. Generally, there are two approaches to study the dynamical behaviour of real physical systems. One is the approximation of the first-order difference equation [11] to physical systems. For classical mechanics in the one-dimensional case, we have easily the first-order nonlinear autonomous equation $x' = \pm[\frac{2}{m}(E - V(x))]^{1/2} \equiv G(x, c)$; $E, V(x)$ are initial energy and potential energy depending on the coordinate x of the particle, c are control parameters. Under the difference approximation their dynamics can be viewed as a mapping form $x_{n+1=}x_n + G(x_n, c)$. When $G(x, c) = a_0x^4 + b_0x^3 + c_0x^2 + d_0x + e_0$, these are standard trimodal maps. They may be the physical models of polynomial potentials and other quartic effective potential models.

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For the optical system, for instance, the periodic and chaotic behaviours come from the instability of the hybrid bistable optical cavity with liquid crystal medium. This leads to the mathematical models of symbolic dynamics of a general trimodal map [12–14], or in the simplified case, the sine-square map is a special one with two equally high peaks [8, 9] in the sense of first-order difference. Second is the Poincaré section method of second-order differential equations. Using this method most physical systems can be modelled as iterative dynamics by the first return map in the Poincaré section [15]. Unfortunately, almost all iterative dynamics in the Poincaré section observed in the numerical study are the multimodal maps with one or more discontinuous points. A typical example is the well known Lorenz equations [15, 16], their iterative dynamics can be modelled by the map of four letters with one discontinuous point [16]. Many physical systems are discontinuous maps such as strongly dissipative systems [17–19]. Although we study the continuous map in this paper, the discontinuous map can be regarded as the breaking or pruning of the continuous map at a region [17–19]. Symbolic dynamics of three-parameter families of general trimodal maps may relate to, for instance, the general circle maps with two non-monotonic intervals [20]. The clearer the continuous map we understand, the deeper the discontinuous map we study. It is our aim to approach real physical systems in this way.

The key rule in symbolic dynamics is the star product. Early on, Derrida *et al* [2] presented a complete expression of the star product for unimodal maps of two letters. Over the past decade some important results have been obtained for bimodal maps of three letters [21–30]. For the one-dimensional symbolic dynamics of continuous maps the next problem to be tackled is the star product of trimodal maps of four letters. Because their dynamical behaviours are generally rather more complex than unimodal and bimodal maps and will display the generic character of continuous maps that unimodal and bimodal ones do not have. For instance, the character of a cyclic group caused by three or more critical points of the map. To date for trimodal maps we have only preliminary results [8–10]. However, seeking their complete start products will be a cumbersome and difficult task mathematically, because the essential rules of the composition operations of period-doubling and period-tripling have not yet been set up. Our work will be parallel to the work of MacKay and Tresser on the bimodal case [21]. It is appropriate to seek some *star transformations*, we describe them in this way as they do not form *complete* star products. Obtaining these star transformations is only the first step to explore the complete star products. After having understood these preliminary rules we will then be able to obtain some clues to the construction of the complete star products, as in the case of bimodal maps [21, 22]. Ringland and Tresser [26, 27] generated a genealogy of finite kneading sequences by using the hierarchical transformations for α-seed, ψ-seeds and χ-seed. They present all monotone equivalence classes of the kneading sequence. In this paper we also generate a genealogy of the kneading sequences with three critical points which form the variety of star transformations by cyclic rules of parity. We present not only the monotone equivalence classes, but also the equal entropy classes which are renormalizable. As the star transformations are not yet complete star products, the monotone equivalence and the equal entropy classes here are only a part of that to be found in the future.

It is revealed that the composition rules of the symbolic sequences in trimodal maps are not trivial generalizations from those of star products in unimodal and bimodal maps. This non-triviality stems from the following. (i) The power sequence W^{*n} of star products (or transformations) of a primitive word W is the formal expression of renormalization. One realizes that there is only one kind of star product (i.e. the Derrida–Gervois–Pomeau $*$ product [2]) for the symbolic dynamics of two letters in unimodal maps and two kinds of dual star products for the symbolic dynamics of three letters in bimodal maps [21, 29, 30].

It is found that there are 3×3 types of rules for period-tripling for triply superstable words in trimodal maps. It is surprising that the number of star transformations, or probably star products, increases exponentially. We conjecture that the number of star transformations is related to the number of cyclic ways which are composed of periodic orbits of critical points. Thus the varieties of star transformations increase the complexity in the renormalization. (ii) A non-trivial generalization in the symbolic dynamics of four letters is the composition operations of period-doubling. One knows that for symbolic dynamics of two or three letters, the rules of period-doubling of a word W are just the *direct* star products of period-2 superstable or doubly superstable words such as $W * RC$ and $W \overline{*} DC$ or $W \underline{*} CD$ [21, 30]. However, for symbolic dynamics of four letters, there are no direct star products for period-doubling. We need to modify the distributive law of multiplications in the star products, or exactly, period-doubling will be realized by new kinds of star transformations. (iii) The increase of the number of star transformations of period-doubling leads to a variety of routes to chaos. An important phenomenon will occur in the Feigenbaum scenario, namely, new Feigenbaum period-doubling bifurcations will possess topological universality but not metric universality (namely, the universal scaling factors and convergent rates disappear) [29].

Summarizing the above, the symbolic dynamics of four letters has many non-trivial features. These non-trivial features would come from the complex permutation of the cyclic group of three critical points, which are not present for the two critical points case. In order to reflect the complexity of the symbolic dynamics of trimodal maps, we introduce some new quantities: the two cyclic permutation operators $\sigma_{I,II}$ of two cyclic components and the directions $S_{C,D,E}$ of parity of the three critical points. The two cyclic ways are anticlockwise and clockwise, and two cyclic components are the three sequences Z, X, Y and critical points E, D, C. The two quantities make the star transformations form a harmonic formalism under cyclic permutation. This expresses the generic characters of the continuous map with multicritical points. It is believable that they also provide a simple prototype to probe the complexity.

This paper is organized as follows. In section 2 the kneading space, word-lifting technique and admissibility conditions of trimodal quartic maps are introduced. In section 3 the star transformations of period-tripling for an arbitrary triply superstable kneading (TSSK) sequence are presented and new mathematical quantities are introduced. The star transformations of period-doubling of TSSK sequences and the rules of period-doubling cascade of arbitrary doubly superstable kneading (DSSK) sequences are discussed in section 4. Finally in section 5 the method to generate singly superstable kneading (SSSK) sequences from TSSK sequences is given by the star transformations.

2. Preliminaries of symbolic dynamics of trimodal map

2.1. Kneading space

Let the horizontal coordinates of three critical points C, D and E be a, b and c respectively. The general quartic map can be written as

$$y = f(x) = -\lambda \left(\frac{1}{4}x^4 - \frac{a+b+c}{3}x^3 + \frac{ab+ac+bc}{2}x^2 - abcx \right) + \delta. \tag{2.1}$$

According to the successive signs of the slope of $f(x)$ on its four intervals of monotonicity, the quartic map can be divided into two types: $(+-+-)$ for $\lambda > 0$, and $(-+-+)$ for $\lambda < 0$. In this paper we only consider the $(+-+-)$ type, the discussion of the $(-+-+)$ type is similar.

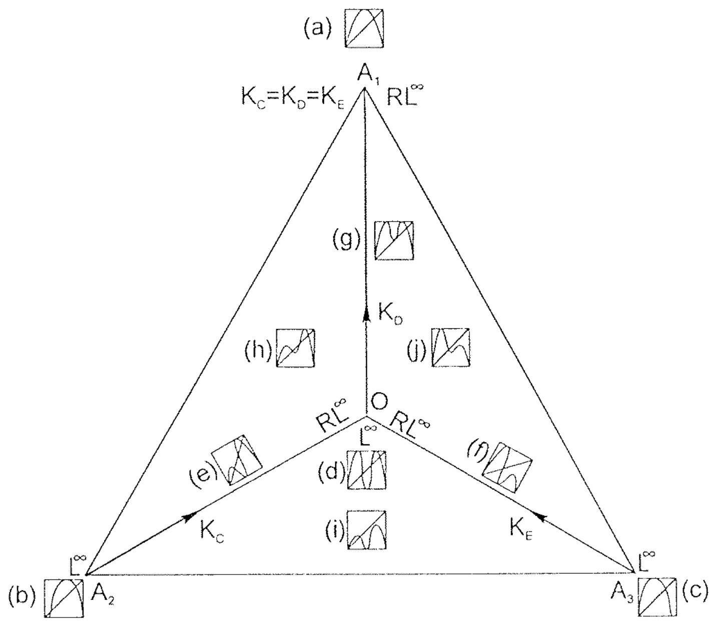

Figure 1. Schematic graphs of maps corresponding to boundary lines and boundary planes in three-dimensional kneading space. Points A_1, A_2, A_3 and O correspond to graphs of maps (a)–(d), respectively. The coordinate axes K_C, K_E and K_D correspond to graphs of maps (e)–(g), respectively. The kneading planes A_1A_2O, A_2A_3O and A_3A_1O corresponding to graphs of maps (h)–(j), respectively.

For the $(+-+-)$ quartic map, the conditions of the interval map should be $f(-1) = -1$, $f(+1) = -1$. The four successive subintervals of monotonicity of the interval $[-1, +1]$ are denoted as L, M, N, and R by turns from left to right with the natural order $L \prec C \prec M \prec D \prec N \prec E \prec R$ where $\prec$ is the MSS order [1] or lexicographical order, L and N are monotone increasing, and M and R monotone decreasing.

The three-dimensional kneading space of words in the symbolic dynamics of four letters is a direct extension of the two-dimensional kneading plane in the symbolic dynamics of three letters [21]. The kneading sequences starting from the three critical points denoted by K_C, K_D and K_E, respectively. Let three coordinate axes correspond to K_C, K_D and K_E, then a three-dimensional space of kneading sequence is formed. The arrows in three axes indicate the directions of increasing in the order of words. The upper and lower boundaries of K_C, K_D and K_E are RL^∞ and L^∞, respectively. The schematic graphs of various maps in the kneading space are given in figure 1.

In order to express the bifurcation structure of three-parameter families of trimodal maps, we need to describe kneading sequences in the three-dimensional space. An arbitrary singly superstable periodic orbit will span a curved surface called the '*membrane surface*' in this space. If the two singly superstable periodic orbits are compatible [8], then their two membrane surfaces will intersect a common curve called the '*bone*'. If the three singly superstable periodic orbits are compatible, then their three membrane surfaces will intersect a common point called the '*joint*'. Of course, the two-dimensional *skeleton* in [21] is extended to the three-dimensional one. Figure 2 gives a sketch of the projections of kneading sequences in the three kneading coordinate planes in which the period of singly superstable periodic orbits is smaller than or equal to 4.

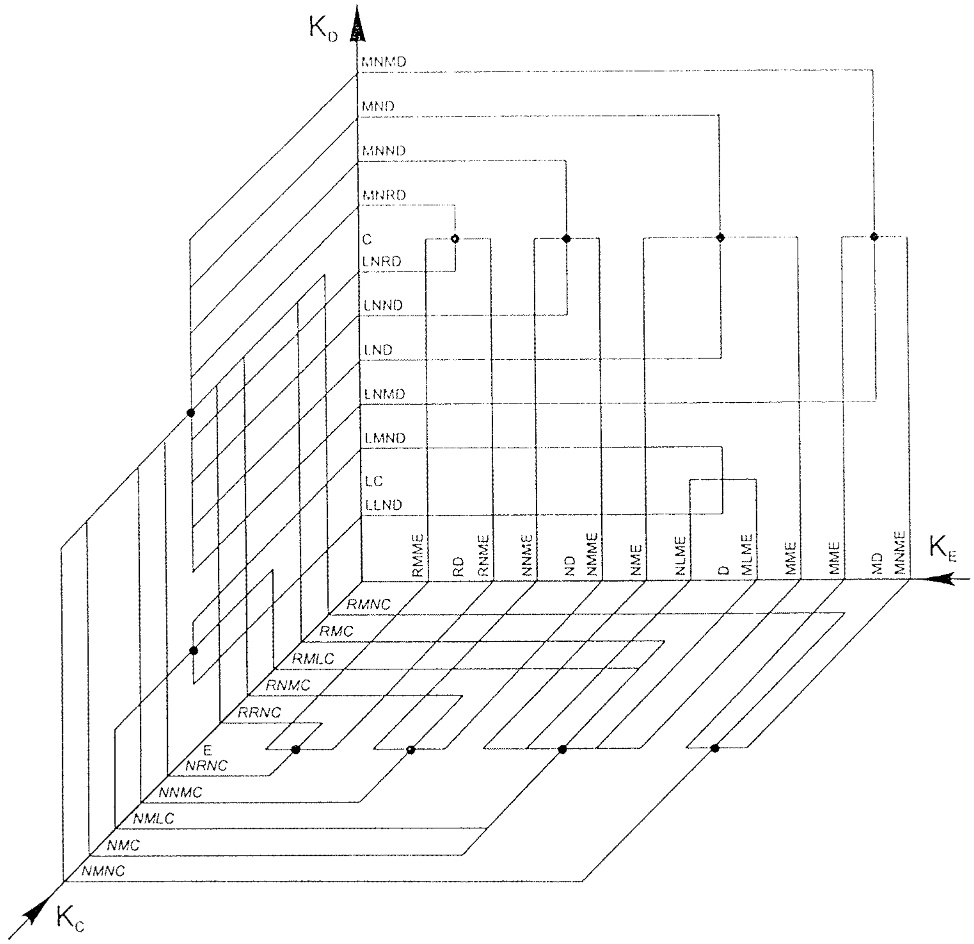

Figure 2. A sketch of the kneading sequences of period smaller than or equal to 4 in the kneading space for the quartic map. The black circles represent projections of the joints to the three kneading coordinate planes.

2.2. Word-lifting technique of quartic maps

In this section the parameters of quartic maps are given by the word-lifting technique [6]. The four inverse functions of the map (2.1) are listed as follows:

$$\begin{aligned}
f_L^{-1} &= \frac{\mu}{2}\left(\sqrt{2t_0}+\sqrt{2t_0-4\left(\frac{p}{2}+t_0+\frac{q}{\sqrt{8t_0}}\right)}\right)+\nu \\
f_M^{-1} &= \frac{\mu}{2}\left(\sqrt{2t_0}-\sqrt{2t_0-4\left(\frac{p}{2}+t_0+\frac{q}{\sqrt{8t_0}}\right)}\right)+\nu \\
f_N^{-1} &= -\frac{\mu}{2}\left(\sqrt{2t_0}-\sqrt{2t_0-4\left(\frac{p}{2}+t_0-\frac{q}{\sqrt{8t_0}}\right)}\right)+\nu \\
f_R^{-1} &= -\frac{\mu}{2}\left(\sqrt{2t_0}+\sqrt{2t_0-4\left(\frac{p}{2}+t_0-\frac{q}{\sqrt{8t_0}}\right)}\right)+\nu
\end{aligned} \tag{2.2}$$

where t_0 is the real positive root of the following cubic equation

$$t^3 + pt^2 + \left(\frac{p^2}{4} + \frac{y - \nu}{\mu} - \gamma\right) t - \frac{q^2}{8} = 0 \tag{2.3}$$

and

$$\nu = \frac{a + b + c}{3} \qquad \mu = \left(-\frac{4}{\lambda}\right)^{1/3} \qquad (\lambda > 0)$$

$$p = -\lambda\mu \left(\frac{3}{2}\nu^2 - (a + b + c)\nu + \frac{ab + ac + bc}{2}\right)$$

$$q = -\lambda(\nu^3 - (a + b + c)\nu^2 + (ab + ac + bc)\nu - abc)$$

$$\gamma = -\frac{\lambda}{\mu} \left(\frac{\nu^4}{4} - \frac{(a + b + c)\nu^3}{3} + \frac{(ab + ac + bc)\nu^2}{2} - abc\nu\right) + \frac{\delta - \nu}{\mu}.$$

Consider a TSSK sequence $(K_C K_E K_D) = (ZEXDYC)$, where $K_C = ZE$, $K_E = XD$ and $K_D = YC$ are the superstable kneading sequences of the critical points C, D and E, respectively. Assume that $ZE = z_1 z_2 \ldots z_n E$, $XD = x_1 x_2 \ldots x_l D$ and $YC = y_1 y_2 \ldots y_m C$ $(z_i, x_i, y_i \in \{L, M, N, R\})$, then the systems of equations of parameters are

$$\begin{aligned} f(a) &= f_{z_1}^{-1} \circ f_{z_2}^{-1} \circ \cdots \circ f_{z_n}^{-1}(c) \\ f(c) &= f_{x_1}^{-1} \circ f_{x_2}^{-1} \circ \cdots \circ f_{x_l}^{-1}(b) \\ f(b) &= f_{y_1}^{-1} \circ f_{y_2}^{-1} \circ \cdots \circ f_{y_m}^{-1}(a). \end{aligned} \tag{2.4}$$

Combining the conditions of the interval map, the relationship between the parameters read

$$b = -\frac{a + c}{1 + 3ac} \qquad \lambda = \frac{4(1 + \delta)}{1 + 2(ab + ac + bc)}. \tag{2.5}$$

Using the iterative method, the parameter values a, c and δ can be calculated. Table 1 lists the values of the kneading parameters for TSSK sequences of period 3–5.

2.3. *Admissibility conditions*

Let $\mathfrak{B}_L$, $\mathfrak{B}_M$, $\mathfrak{B}_N$ and $\mathfrak{B}_R$ be the sets of all subsequences which follow L, M, N and R, respectively. Assume that W is an admissible word, then the admissibility conditions are as follows [31]:

$$\begin{aligned} &\mathfrak{B}_L(W) \leqslant K_C \qquad K_D \leqslant \mathfrak{B}_M(W) \leqslant K_C \\ &K_D \leqslant \mathfrak{B}_N(W) \leqslant K_E \qquad \mathfrak{B}_R(W) \leqslant K_E. \end{aligned} \tag{2.6}$$

When K_C, K_D and K_E are superstable kneading sequences, the admissibility conditions turn into pure inequalities. Obviously the kneading sequences themselves should also satisfy the admissibility conditions

$$\begin{aligned} &\mathfrak{B}_L(K_C, K_D, K_E) \leqslant K_C \qquad K_D \leqslant \mathfrak{B}_M(K_C, K_D, K_E) \leqslant K_C \\ &K_D \leqslant \mathfrak{B}_N(K_C, K_D, K_E) \leqslant K_E \qquad \mathfrak{B}_R(K_C, K_D, K_E) \leqslant K_E. \end{aligned} \tag{2.7}$$

3. Varieties of star transformations of period-tripling of TSSK sequences

The method to produce TSSK sequences can be obtained by the star transformations of three critical points. The periodic points of TSSK sequences of finite period belong to the cyclic group which possesses two different types of cyclic ways, $(K_C K_E K_D) = (ZEXDYC)$ and $(K_D K_E K_C) = (ZEYCXD)$. According to the admissibility conditions we find that TSSK

Table 1. The values of kneading parameters of the $(ZEXDYC)$ TSSK sequences of period 3–5 for the quartic map.

Period	Sequence	a	c	δ
3	*EDC*	−0.663 307 953 129 798	0.699 930 313 522 219	−0.624 636 883 352 953
4	*EDLC*	−0.685 205 996 622 996	0.726 195 072 264 293	−0.936 339 412 433 705
4	*EMDC*	−0.631 287 900 353 161	0.702 979 509 029 810	−0.482 208 991 407 285
4	*ENDC*	−0.679 940 949 756 513	0.693 291 406 985 949	−0.674 449 665 297 450
4	*ERDC*	−0.696 078 325 316 184	0.685 049 377 798 136	−0.691 965 475 804 164
5	*EDLMC*	−0.681 569 729 522 127	0.721 947 208 280 861	−0.878 365 779 471 217
5	*EDLNC*	−0.677 470 890 386 795	0.717 106 391 666 351	−0.816 175 103 863 965
5	*EMDLC*	−0.661 066 787 854 674	0.750 788 190 950 871	−0.825 547 972 881 237
5	*ENDLC*	−0.699 400 077 125 028	0.712 199 589 734 198	−0.971 917 481 788 540
5	*ERDLC*	−0.711 398 643 362 396	0.700 676 538 205 524	−0.974 524 234 989 478
5	*EMMDC*	−0.648 330 548 452 229	0.703 444 351 691 021	−0.562 930 439 711 044
5	*ENMDC*	−0.674 547 908 194 458	0.695 690 898 527 780	−0.661 117 634 285 177
5	*ENNDC*	−0.684 351 860 393 684	0.691 183 065 063 070	−0.682 880 817 315 341
5	*ERLDC*	−0.698 280 398 525 304	0.683 824 882 873 100	−0.691 111 395 665 345
5	*ERMDC*	−0.697 114 566 288 067	0.684 475 702 925 334	−0.691 676 546 661 112
5	*ERNDC*	−0.694 627 041 376 425	0.685 844 971 947 333	−0.692 043 562 350 866
5	*NERDC*	−0.699 646 418 753 917	0.680 110 547 464 113	−0.687 244 764 652 366
5	*RERDC*	−0.693 500 637 773 340	0.688 583 929 722 396	−0.692 658 892 138 212
5	*RMEDC*	−0.660 601 489 396 589	0.706 537 548 164 126	−0.596 231 780 232 299

sequences have three different kinds of generating rules. By the cyclic operation, we find that the star transformations of critical points in dynamical systems of four letters have nine rules for each TSSK sequence. The varieties of star transformations of TSSK sequences imply that symbolic dynamics of trimodal or multimodal maps is rather more complex than that of unimodal or bimodal maps.

Let $\mathfrak{B}_{L,M}$ stand for $\mathfrak{B}_L \cup \mathfrak{B}_M$, $\mathfrak{B}_{M,N}$ for $\mathfrak{B}_M \cup \mathfrak{B}_N$, $\mathfrak{B}_{N,R}$ for $\mathfrak{B}_N \cup \mathfrak{B}_R$. We define two cyclic permutation operators of two cyclic components as

$$\sigma_I = \begin{pmatrix} Z & X & Y & , & E & D & C \\ Y & Z & X & , & C & E & D \end{pmatrix} \qquad \sigma_{II} = \begin{pmatrix} Z & X & Y & , & E & D & C \\ X & Y & Z & , & D & C & E \end{pmatrix} \tag{3.1}$$

where σ_I is anticlockwise and σ_{II} clockwise cyclic, the sequences (Z, X and Y) and critical points (E, D and C) form two components. Let $S_E = \pm$, $S_D = \pm$ and $S_C = \pm$, which represent the directions of parity. $S = +$ is called the forward parity which implies the upper sequence of the window, $S = -$ the backward parity which implies the lower sequence of the window.

Lemma 1. If $(K_C K_E K_D) = (ZEXDYC)$ is a TSSK sequence of arbitrary period, then for the anticlockwise cyclic permutation operation, the following results hold

(I) $\mathfrak{B}_{L,M}(\sigma_I^j(ZE^{S_E\tau(Z)}XD^{S_D\tau(X)}YC^{S_C\tau(Y)}ZE))$

$$< \begin{cases} ZE^{+\tau(Z)}XD^{-\tau(X)}YC^{+\tau(Y)} & \text{if } S_E = + \\ ZE & \text{if } S_E = - \end{cases} \tag{3.2a}$$

(II) $\mathfrak{B}_{M,,N}(\sigma_I^j(ZE^{S_E\tau(Z)}XD^{S_D\tau(X)}YC^{S_C\tau(Y)}ZE))$

$$> \begin{cases} YC & \text{if } S_C = + \\ YC^{-\tau(Y)}ZE^{-\tau(Z)}XD^{-\tau(X)} & \text{if } S_C = - \end{cases} \tag{3.2b}$$

(III) $\mathfrak{B}_{N,R}(\sigma_I^j(ZE^{S_E\tau(Z)}XD^{S_D\tau(X)}YC^{S_C\tau(Y)}ZE))$

$$< \begin{cases} XD^{+\tau(X)}YC^{+\tau(Y)}ZE^{-\tau(Z)} & \text{if } S_D = + \\ XD & \text{if } S_D = - \end{cases} \tag{3.2c}$$

where j represents the number of cyclic permutations, and $j = 0, 1, 2$ corresponding to $S_E = +$, $S_C = -$, $S_D = +$, respectively. The parity inverse operator τ for $W \in \{Z, X, Y\}$ is defined as

$$\tau(W) = \begin{cases} + & \text{if } W \text{ is even} \\ - & \text{if } W \text{ is odd} \end{cases} \tag{3.3}$$

we say W is *even* if it contains an even number of M's and R's, and *odd* otherwise.

Now we give only a concise explanation for (I) in lemma 1. If $j = 0$, (I) becomes

$$\mathfrak{B}_{L,M}(ZE^{+(Z)}XD^{\pm\tau(X)}YC^{\pm\tau(Y)}ZE) < ZE^{+(Z)}XD^{-\tau(X)}YC^{+\tau(Y)}. \tag{3.4}$$

If $j = 1$, (I) becomes

$$\mathfrak{B}_{L,M}(YC^{-\tau(Y)}ZE^{+\tau(Z)}XD^{\pm\tau(X)}YC) < ZE^{+(Z)}XD^{-\tau(X)}YC^{+\tau(Y)} \tag{3.5a}$$

$$\mathfrak{B}_{L,M}(YC^{-\tau(Y)}ZE^{-\tau(Z)}XD^{\pm\tau(X)}YC) < ZE. \tag{3.5b}$$

If $j = 2$, (I) becomes

$$\mathfrak{B}_{L,M}(XD^{+\tau(X)}YC^{\pm\tau(Y)}ZE^{+\tau(Z)}XD) < ZE^{+(Z)}XD^{-\tau(X)}YC^{+\tau(Y)} \tag{3.6a}$$

$$\mathfrak{B}_{L,M}(XD^{+\tau(X)}YC^{\pm\tau(Y)}ZE^{-\tau(Z)}XD) < ZE.(3.6b) \tag{3.6b}$$

Because $(K_C K_E K_D) = (ZEXDYC)$, its admissibility conditions read

$$\mathfrak{B}_{L,M}(ZE) < ZE \qquad \mathfrak{B}_{L,M}(XD) < ZE \qquad \mathfrak{B}_{L,M}(YC) < ZE \tag{3.7a}$$

$$\mathfrak{B}_{M,N}(ZE) > YC \qquad \mathfrak{B}_{M,N}(XD) > YC \qquad \mathfrak{B}_{M,N}(YC) > YC \tag{3.7b}$$

$$\mathfrak{B}_{N,R}(ZE) < XD \qquad \mathfrak{B}_{N,R}(XD) < XD \qquad \mathfrak{B}_{N,R}(YC) < XD. \tag{3.7c}$$

By using the above admissibility conditions the proof of (I) is presented in appendix A, and the proofs of (II) and (III) are similar.

Lemma 2. If $(K_D K_E K_C) = (ZEYCXD)$ is a TSSK sequence of arbitrary period, then for the clockwise cyclic permutation operation, we have

(I) $\mathfrak{B}_{L,M}(\sigma_{II}^{j}(ZE^{S_E\tau(Z)}YC^{S_C\tau(Y)}XD^{S_D\tau(X)}ZE))$

$$< \begin{cases} XD^{+\tau(X)}ZE^{+\tau(Z)}YC^{-\tau(Y)} & \text{if } S_D = + \\ XD & \text{if } S_D = - \end{cases} \tag{3.8a}$$

(II) $\mathfrak{B}_{M,N}(\sigma_{II}^{j}(ZE^{S_E\tau(Z)}YC^{S_C\tau(Y)}XD^{S_D\tau(X)}ZE))$

$$> \begin{cases} ZE & \text{if } S_E = + \\ ZE^{-\tau(Z)}YC^{-\tau(Y)}XD^{-\tau(X)} & \text{if } S_E = - \end{cases} \tag{3.8b}$$

(III) $\mathfrak{B}_{N,R}(\sigma_{II}^{j}(ZE^{S_E\tau(Z)}YC^{S_C\tau(Y)}XD^{S_D\tau(X)}ZE))$

$$< \begin{cases} YC^{+\tau(Y)}XD^{-\tau(X)}ZE^{+\tau(Z)} & \text{if } S_C = + \\ YC & \text{if } S_C = - \end{cases} \tag{3.8c}$$

where $j = 0, 1, 2$ corresponding to $S_E = -$, $S_D = +$, $S_C = +$, respectively.

The proof of lemma 2 is similar to that of lemma 1 by using the following admissibility conditions:

$$\mathfrak{B}_{L,M}(ZE) < XD \qquad \mathfrak{B}_{L,M}(YC) < XD \qquad \mathfrak{B}_{L,M}(XD) < XD \qquad (3.9a)$$
$$\mathfrak{B}_{M,N}(ZE) > ZE \qquad \mathfrak{B}_{M,N}(YC) > ZE \qquad \mathfrak{B}_{M,N}(XD) > ZE \qquad (3.9b)$$
$$\mathfrak{B}_{N,R}(ZE) < YC \qquad \mathfrak{B}_{N,R}(YC) < YC \qquad \mathfrak{B}_{N,R}(XD) < YC. \qquad (3.9c)$$

Suppose that the star transformations satisfy the following distributive law of left multiplication

$$(ZEXDYC) * (EDC) = (ZEXDYC) * E(ZEXDYC) * D(ZEXDYC) * C$$
$$= ZEXD^{S_D\tau(X)}YC^{S_C\tau(Y)}ZE^{S_E\tau(Z)}XDYC^{S_C'\tau(Y)}ZE^{S_E'\tau(Z)}XD^{S_D'\tau(X)}YC$$

where S_D S_C, S_E, S_D', S_C' and S_E' are signs of parity directions which take $+$ or $-$ depending on the different types of star transformations.

For the generating rules of period-tripling of TSSK sequences, we have the following theorem.

Theorem 1. If $(K_C K_E K_D) = (ZEXDYC)$ is an arbitrary TSSK sequence, then the period-tripling transformations will generate the following TSSK sequences:

(I) $\mathfrak{T}_1(\sigma_I^0(ZEXDYC)) := (ZEXDYC) *_{1,I^0} (EDC)$
$$= ZEXD^{+\tau(X)}YC^{-\tau(Y)}ZE^{-\tau(Z)}XDYC^{-\tau(Y)}ZE^{-\tau(Z)}XD^{-\tau(X)}YC \qquad (3.10a)$$
$\mathfrak{T}_1(\sigma_I^1(ZEXDYC)) := (YCZEXD) *_{1,I^1} (CED)$
$$= YCZE^{+\tau(Z)}XD^{+\tau(X)}YC^{+\tau(Y)}ZEXD^{+\tau(X)}YC^{+\tau(Y)}ZE^{-\tau(Z)}XD \qquad (3.10b)$$
$\mathfrak{T}_1(\sigma_I^2(ZEXDYC)) := (XDYCZE) *_{1,I^2} (DCE)$
$$= XDYC^{-\tau(Y)}ZE^{+\tau(Z)}XD^{-\tau(X)}YCZE^{+\tau(Z)}XD^{-\tau(X)}YC^{+\tau(Y)}ZE \qquad (3.10c)$$

(II) $\mathfrak{T}_2(\sigma_I^0(ZEXDYC)) := (ZEXDYC) *_{2,I^0} (EDC)$
$$= ZEXD^{+\tau(X)}YC^{+\tau(Y)}ZE^{-\tau(Z)}XDYC^{-\tau(Y)}ZE^{-\tau(Z)}XD^{-\tau(X)}YC \qquad (3.11a)$$
$\mathfrak{T}_2(\sigma_I^1(ZEXDYC)) := (YCZEXD) *_{2,I^1} (CED)$
$$= YCZE^{+\tau(Z)}XD^{-\tau(X)}YC^{+\tau(Y)}ZEXD^{+\tau(X)}YC^{+\tau(Y)}ZE^{-\tau(Z)}XD \qquad (3.11b)$$
$\mathfrak{T}_2(\sigma_I^2(ZEXDYC)) := (XDYCZE) *_{2,I^2} (DCE)$
$$= XDYC^{-\tau(Y)}ZE^{-\tau(Z)}XD^{-\tau(X)}YCZE^{+\tau(Z)}XD^{-\tau(X)}YC^{+\tau(Y)}ZE \qquad (3.11c)$$

(III) $\mathfrak{T}_3(\sigma_I^0(ZEXDYC)) := (ZEXDYC) *_{3,I^0} (EDC)$
$$= ZEXD^{+\tau(X)}YC^{+\tau(Y)}ZE^{-\tau(Z)}XDYC^{-\tau(Y)}ZE^{-\tau(Z)}XD^{+\tau(X)}YC \qquad (3.12a)$$
$\mathfrak{T}_3(\sigma_I^1(ZEXDYC)) := (YCZEXD) *_{3,I^1} (CED)$
$$= YCZE^{+\tau(Z)}XD^{-\tau(X)}YC^{+\tau(Y)}ZEXD^{+\tau(X)}YC^{+\tau(Y)}ZE^{+\tau(Z)}XD \qquad (3.12b)$$
$\mathfrak{T}_3(\sigma_I^2(ZEXDYC)) := (XDYCZE) *_{3,I^2} (DCE)$
$$= XDYC^{-\tau(Y)}ZE^{-\tau(Z)}XD^{-\tau(X)}YCZE^{+\tau(Z)}XD^{-\tau(X)}YC^{-\tau(Y)}ZE. \qquad (3.12c)$$

Theorem 1 can be proved from lemma 1 and admissibility conditions (3.7*a*–*c*).

Theorem 2. For another type of TSSK sequence $(K_D K_E K_C) = (ZEYCXD)$, the period-tripling transformations will generate the following TSSK sequences:

(I) $\mathfrak{T}_1(\sigma_{II}^0(ZEYCXD)) := (ZEYCXD) *_{1,II^0} (ECD)$
$$= ZEYC^{+\tau(Y)}XD^{-\tau(X)}ZE^{+\tau(Z)}YCXD^{+\tau(X)}ZE^{+\tau(Z)}YC^{+\tau(Y)}XD \qquad (3.13a)$$

$$\mathfrak{T}_1(\sigma_{II}^1(ZEYCXD)) := (XDZEYC) *_{1,II^1} (DEC)$$
$$= XDZE^{-\tau(Z)}YC^{+\tau(Y)}XD^{-\tau(X)}ZEYC^{+\tau(Y)}XD^{-\tau(X)}ZE^{+\tau(Z)}YC \qquad (3.13b)$$
$$\mathfrak{T}_1(\sigma_{II}^2(ZEYCXD)) := (YCXDZE) *_{1,II^2} (CDE)$$
$$= YCXD^{+\tau(X)}ZE^{-\tau(Z)}YC^{-\tau(Y)}XDZE^{-\tau(Z)}YC^{-\tau(Y)}XD^{-\tau(X)}ZE \qquad (3.13c)$$

(II) $$\mathfrak{T}_2(\sigma_{II}^0(ZEYCXD)) := (ZEYCXD) *_{2,II^0} (ECD)$$
$$= ZEYC^{+\tau(Y)}XD^{-\tau(X)}ZE^{+\tau(Z)}YCXD^{+\tau(X)}ZE^{+\tau(Z)}YC^{-\tau(Y)}XD \qquad (3.14a)$$
$$\mathfrak{T}_2(\sigma_{II}^1(ZEYCXD)) := (XDZEYC) *_{2,II^1} (DEC)$$
$$= XDZE^{-\tau(Z)}YC^{-\tau(Y)}XD^{-\tau(X)}ZEYC^{+\tau(Y)}XD^{-\tau(X)}ZE^{+\tau(Z)}YC \qquad (3.14b)$$
$$\mathfrak{T}_2(\sigma_{II}^2(ZEYCXD)) := (YCXDZE) *_{2,II^2} (CDE)$$
$$= YCXD^{+\tau(X)}ZE^{+\tau(Z)}YC^{-\tau(Y)}XDZE^{-\tau(Z)}YC^{-\tau(Y)}XD^{-\tau(X)}ZE \qquad (3.14c)$$

(III) $$\mathfrak{T}_3(\sigma_{II}^0(ZEYCXD)) := (ZEYCXD) *_{3,II^0} (ECD)$$
$$= ZEYC^{+\tau(Y)}XD^{+\tau(X)}ZE^{+\tau(Z)}YCXD^{+\tau(X)}ZE^{+\tau(Z)}YC^{-\tau(Y)}XD \qquad (3.15a)$$
$$\mathfrak{T}_3(\sigma_{II}^1(ZEYCXD)) := (XDZEYC) *_{3,II^1} (DEC)$$
$$= XDZE^{-\tau(Z)}YC^{-\tau(Y)}XD^{-\tau(X)}ZEYC^{+\tau(Y)}XD^{-\tau(X)}ZE^{-\tau(Z)}YC \qquad (3.15b)$$
$$\mathfrak{T}_3(\sigma_{II}^2(ZEYCXD)) := (YCXDZE) *_{3,II^2} (CDE)$$
$$= YCXD^{+\tau(X)}ZE^{+\tau(Z)}YC^{-\tau(Y)}XDZE^{-\tau(Z)}YC^{-\tau(Y)}XD^{+\tau(X)}ZE. \qquad (3.15c)$$

Theorem 2 can be similarly proved from lemma 2 and admissibility conditions (3.9*a*–*c*).

In theorems 1 and 2, we have 3×3 kinds of star transformations for each of the cyclic ways in the period-tripling. They display the variety of types of star transformations. It should be indicated that the TSSK sequences generated by the period-tripling transformations in (3.10*a*)–(3.15*c*) are all star products themselves.

4. Star transformations of period-doubling

4.1. Period-doubling transformations of TSSK sequences

In symbolic dynamics of two or three letters, the method to generate period-doubling sequences is by using the star products of an arbitrary (doubly) superstable word and a period-2 (doubly) superstable word. However, in symbolic dynamics of four letters, this rule is not completely valid, a new generating rule is required. From the admissibility conditions and the forms of period-doubling, we can see that the generating rules of period-tripling of TSSK sequences and the period-doubling are mutually connected. We have the following theorem.

Theorem 3. Suppose that $(K_C K_E K_D) = (ZEXDYC)$ is an arbitrary TSSK sequence, the period-doubling transformations generate the following TSSK sequences:

$$\mathfrak{D}(\sigma_I^0(ZEXDYC)) := ZE(XDYC) *_{2,I^0} E(ZEXD) *_{2,I^0} DYC$$
$$= ZEXD^{+\tau(X)}YC^{+\tau(Y)}ZE^{-\tau(Z)}XDYC \qquad (4.1a)$$
$$\mathfrak{D}(\sigma_I^1(ZEXDYC)) := YC(ZEXD) *_{2,I^1} C(YCZE) *_{2,I^1} EXD$$
$$= YCZE^{+\tau(Z)}XD^{-\tau(X)}YC^{+\tau(Y)}ZEXD \qquad (4.1b)$$
$$\mathfrak{D}(\sigma_I^2(ZEXDYC)) := XD(YCZE) *_{2,I^2} D(XDYC) *_{2,I^2} CZE$$
$$= XDYC^{-\tau(Y)}ZE^{-\tau(Z)}XD^{-\tau(X)}YCZE. \qquad (4.1c)$$

The proof of theorem 3 depends on the admissibility conditions (3.7*a*–*c*). Using lemma 1 one can prove the theorem.

Theorem 4. Suppose that $(K_D K_E K_C) = (ZEYCXD)$ is an arbitrary TSSK sequence, the period-doubling transformations generate the following TSSK sequences:

$$\mathfrak{D}(\sigma_{II}^0(ZEYCXD)) := ZE(YCXD) *_{2,II^0} E(ZEYC) *_{2,II^0} CXD$$
$$= ZEYC^{+\tau(Y)}XD^{-\tau(X)}ZE^{+\tau(Z)}YCXD \qquad (4.2a)$$

$$\mathfrak{D}(\sigma_{II}^1(ZEYCXD)) := XD(ZEYC) *_{2,II^1} D(XDZE) *_{2,II^1} EYC$$
$$= XDZE^{-\tau(Z)}YC^{-\tau(Y)}XD^{-\tau(X)}ZEYC \qquad (4.2b)$$

$$\mathfrak{D}(\sigma_{II}^2(ZEYCXD)) := YC(XDZE) *_{2,II^2} C(YCXD) *_{2,II^2} DZE$$
$$= YCXD^{+\tau(X)}ZE^{+\tau(Z)}YC^{-\tau(Y)}XDZE. \qquad (4.2c)$$

Proof of theorem 4 depends on the admissibility conditions (3.9*a*–*c*). Using lemma 2 one can prove the theorem.

The generating rules of period-doubling have the parity symmetry. For period-doubling of TSSK sequences in theorems 3 and 4 we see that S_D takes the same parity, while S_E (or S_C) takes opposite parity among the expressions (4.1*a*) and (4.2*c*), (4.1*b*) and (4.2*b*), as well as (4.1*c*) and (4.2*a*). Thus $S_E S_D S_C$ remains unchanged in each pair of expressions. Similarly, the preservation of parity symmetry occurs in the next paragraph. For period-doubling of DSSK sequences in (I) of theorem 5, we also see that S_C and S_D in (4.4*a*) have opposite signs to those in (4.4*b*), so $S_C S_D$ remains unchanged; while in (4.5*a*) and (4.5*b*) of (II), $S_C S_E$ remains unchanged.

4.2. Period-doubling transformations of DSSK sequences and cascades

The generating rule of period-doubling of DSSK sequences in the symbolic dynamics of three letters is obtained on the basis of the up-star and down-star products of sequences of critical points [21, 30] that in the symbolic dynamics of four letters can also be found by the star transformations of sequences of critical points. One can also divide DSSK sequences of four letters into two types of cyclic ways, $(K_E|K_C K_D) = (ZE|XDYC)$ and $(K_D|K_C K_E) = (XD|ZEYC)$. Define two new cyclic permutation operators of two cyclic components as

$$\sigma_{III} = \begin{pmatrix} Z & X & Y & , & E & D & C \\ Z & Y & X & , & E & C & D \end{pmatrix} \qquad \sigma_{IV} = \begin{pmatrix} Z & X & Y & , & E & D & C \\ Y & X & Z & , & C & D & E \end{pmatrix}. \qquad (4.3)$$

Then we have the following generating rule of period-doubling or star transformations.

Theorem 5. Suppose that $(K_E|K_C K_D) = (ZE|XDYC)$ and $(K_D|K_C K_E) = (XD|ZEYC)$ are arbitrary DSSK sequences, the period-doubling transformations generate the following DSSK sequences:

(I) $\mathfrak{D}(\sigma_{III}^0(ZE|XDYC)) := (ZE|XDYC) *_{1,I^0} (E|DC) = (ZE|XDYC^{-\tau(Y)}XD^{-\tau(X)}YC)$ (4.4*a*)

$\mathfrak{D}(\sigma_{III}^1(ZE|XDYC)) := (ZE|YCXD) *_{1,I^1} (E|CD) = (ZE|YCXD^{+\tau(X)}YC^{+\tau(Y)}XD)$ (4.4*b*)

(II) $\mathfrak{D}(\sigma_{IV}^0(XD|ZEYC)) := (XD|ZEYC) *_{3,I^0} (D|EC) = (XD|ZEYC^{+\tau(Y)}ZE^{-\tau(Z)}YC)$ (4.5*a*)

$\mathfrak{D}(\sigma_{IV}^1(XD|ZEYC)) := (XD|YCZE) *_{3,I^2} (D|CE) = (XD|YCZE^{+\tau(Z)}YC^{-\tau(Y)}ZE).$ (4.5*b*)

Admissibility conditions of (I) in theorem 5 read

$$\mathfrak{B}_{L,M}(ZE) < XD \qquad \mathfrak{B}_{L,M}(XD) < XD \qquad \mathfrak{B}_{L,M}(YC) < XD$$
$$\mathfrak{B}_{M,N}(ZE) > YC \qquad \mathfrak{B}_{M,N}(XD) > YC \qquad \mathfrak{B}_{M,N}(YC) > YC$$
$$\mathfrak{B}_{N,R}(ZE) < ZE \qquad \mathfrak{B}_{N,R}(XD) < ZE \qquad \mathfrak{B}_{N,R}(YC) < ZE.$$

Admissibility conditions of (II) in theorem 5 are

$$\mathfrak{B}_{L,M}(ZE) < ZE \qquad \mathfrak{B}_{L,M}(XD) < ZE \qquad \mathfrak{B}_{L,M}(YC) < ZE$$
$$\mathfrak{B}_{M,N}(ZE) > XD \qquad \mathfrak{B}_{M,N}(XD) > XD \qquad \mathfrak{B}_{M,N}(YC) > XD$$
$$\mathfrak{B}_{N,R}(ZE) < YC \qquad \mathfrak{B}_{N,R}(XD) < YC \qquad \mathfrak{B}_{N,R}(YC) < YC.$$

Theorem 5 can be easily proved by the admissibility conditions and the similar method to prove lemma 1.

For a given DSSK sequence $(K_E|K_C K_D) = (ZE|XDYC)$, where $K_D = YCXD$ and $K_C = XDYC$. The upper and lower sequences of K_D are SSSK sequences $YC^{+\tau(Y)}XD$ and $YC^{-\tau(Y)}XD$. The upper sequence of $YC^{+\tau(Y)}XD$ is $YC^{+\tau(Y)}XD^{-\tau(X)}$, the lower sequence of $YC^{-\tau(Y)}XD$ is $YC^{-\tau(Y)}XD^{-\tau(X)}$. They form the boundaries of the window of K_D. Thus the upper and lower boundaries of period-doubling sequence $YC^{-\tau(Y)}XD^{-\tau(X)}YCXD$ of K_D are $YC^{-\tau(Y)}XD^{-\tau(X)}YC^{-\tau(Y)}XD^{-\tau(X)}$ and $YC^{-\tau(Y)}XD^{-\tau(X)}YC^{+\tau(Y)}XD^{-\tau(X)}$. Similarly, the upper and lower sequences of K_C are SSSK sequences $XD^{+\tau(X)}YC$ and $XD^{-\tau(X)}YC$. The upper and lower boundaries of the window of K_C are $XD^{+\tau(X)}YC^{+\tau(Y)}$ and $XD^{-\tau(X)}YC^{+\tau(Y)}$. The upper and lower boundaries of period-doubling sequence $XD^{+\tau(X)}YC^{+\tau(Y)}XDYC$ of K_C are $XD^{+\tau(X)}YC^{+\tau(Y)}\ XD^{-\tau(X)}YC^{+\tau(Y)}$ and $XD^{+\tau(X)}YC^{+\tau(Y)}XD^{+\tau(X)}YC^{+\tau(Y)}$. Therefore the period-doubling bifurcations occur at odd strings $YC^{-\tau(Y)}XD^{-\tau(X)}$ and $XD^{+\tau(X)}YC^{+\tau(Y)}$. The lower boundary of K_D attaches to the upper one of the period-doubling sequence of K_D. The upper boundary of K_C attaches to the lower one of the period-doubling sequence of K_C. The lower boundary of K_C attaches to the upper one of K_D.

For a given DSSK sequence $(K_D|K_C K_E) = (XD|ZEYC)$, where $K_C = ZEYC$ and $K_E = YCZE$. The upper and lower boundaries of the window of K_C are $ZE^{+\tau(Z)}YC^{-\tau(Y)}$ and $ZE^{-\tau(Z)}YC^{-\tau(Y)}$. The upper and lower boundaries of period-doubling sequence $ZE^{+\tau(Z)}YC^{-\tau(Y)}ZEYC$ of K_C are $ZE^{+\tau(Z)}YC^{-\tau(Y)}ZE^{-\tau(Z)}YC^{-\tau(Y)}$ and $ZE^{+\tau(Z)}YC^{-\tau(Y)}ZE^{+\tau(Z)}YC^{-\tau(Y)}$. Similarly, the upper and lower boundaries of the window of K_E are $YC^{+\tau(Y)}ZE^{-\tau(Z)}$ and $YC^{-\tau(Y)}ZE^{-\tau(Z)}$. The upper and lower boundaries of period-doubling sequence $YC^{+\tau(Y)}ZE^{-\tau(Z)}YCZE$ of K_E are $YC^{+\tau(Y)}ZE^{-\tau(Z)}YC^{-\tau(Y)}ZE^{-\tau(Z)}$ and $YC^{+\tau(Y)}ZE^{-\tau(Z)}YC^{+\tau(Y)}ZE^{-\tau(Z)}$. Therefore the period-doubling bifurcations occur at odd strings $ZE^{+\tau(Z)}YC^{-\tau(Y)}$ and $YC^{+\tau(Y)}ZE^{-\tau(Z)}$. The lower boundaries of K_C and K_E connects with each other, and their upper boundaries attach to the lower boundaries of their period-doubling sequences, respectively. Thus the Feigenbaum period-doubling bifurcation will form a sequences of cascades.

From the generating rule of DSSK sequences, we can easily write the sequences of period-doubling cascade

$$\mathfrak{D}^{*_{1,I0}n}(E|DC) \qquad \mathfrak{D}^{*_{1,I1}n}(E|CD)$$
$$\mathfrak{D}^{*_{3,I0}n}(D|EC) \qquad \mathfrak{D}^{*_{3,I2}n}(D|CE).$$

For example, the beginnings of the sequences ($n = 1, 2, 3, 4 \ldots$) are

$$\mathfrak{D}^{*_{1,I0}n}(E|DC) = (E|DC),\ (E|DLMC),\ (E|DLMMMLMC),$$
$$(E|DLMMMLMLMLMMMLMC), \ldots,$$

$\mathfrak{D}^{*_{1,I^1}n}(E|CD) = (E|CD),\ (E|NMDC),\ (E|NMMMNMDC),$
$(E|NMMMNMNMNMMMNMDC), \ldots,$
$\mathfrak{D}^{*_{3,I^0}n}(D|EC) = (D|EC),\ (D|EMNC),\ (D|EMNLNMNC),$
$(D|EMNLNMNMNMNLNMNC), \ldots,$
$\mathfrak{D}^{*_{3,I^2}n}(D|CE) = (D|CE),\ (D|RLEC),\ (D|RLNLRLEC),$
$(D|RLNLRLRLRLNLRLEC), \ldots.$

Of course the sequences of zero topological entropy can be verified by the Milnor–Thusrston characteristic polynomial [31, 32]. Taking the limit of $n \to \infty$, the boundary of zero topological entropy (i.e. boundary of topological chaos [22]) can be obtained. The boundary will be the complex fractal curved surface in the three-dimensional kneading space of parameters. In addition, we can also see other quite complex fractal objects when taking the limit of power of the various star transformations for the period-tripling, as in the known symbolic dynamics [28, 33, 34]. Now, turning to the phase space to discuss the above example, we can obtain the regular fractal object with a constant fractal dimension, because the power sequence of the star transformations is taken from only one of four kinds of transformations ($\mathfrak{D}^{*_{1,I^0}n_1}$, $\mathfrak{D}^{*_{1,I^1}n_2}$, $\mathfrak{D}^{*_{3,I^0}n_3}$ and $\mathfrak{D}^{*_{3,I^2}n_4}$) namely, the transformations are pure. If the transformations are mixed, then taking from one pair of four kinds of transformations ($\mathfrak{D}^{*_{1,I^0}n_1}$, $\mathfrak{D}^{*_{1,I^1}n_2}$) or ($\mathfrak{D}^{*_{3,I^0}n_3}$, $\mathfrak{D}^{*_{3,I^2}n_4}$) respectively; the limit of the power sequence may be an irregular multifractal object. Moreover, if the power sequence n_1, n_2, (or n_3, n_4) ($\{n_1, n_2, n_3, n_4\} \in Z_+$) is taken as a pseudorandom or random sequence and the limit exists according to the probability convergence, then an interesting phenomenon will occur in the Feigenbaum's scenario, namely, a new Feigenbaum period-doubling bifurcation will possess the topological universality, because the symbolic sequence will preserve the topological universality in the sense of a monotone equivalence class of maps. But the metric universality will no longer be preserved, because the power of symbolic sequence is pseudorandom or random. This leads to the disappearance of the universal scaling factor and convergent rate [29]. The details of this interesting phenomenon will be discussed elsewhere.

In the theorems 3 and 4, we have three kinds of star transformations for each of the cyclic ways in the period-doubling for TSSK sequences, which may be determined by the number of parameters of the maps. In theorem 5 we have two kinds of transformations for DSSK sequences. They display a variety of genealogy of star transformations.

5. The method to generate SSSK sequences from TSSK sequences by star transformations

An important method in symbolic dynamics of four letters to produce all the finite SSSK sequences from TSSK sequences is similar to that of the symbolic dynamics of three letters [26]. The star transformations can play a role. Since there exist three types of star transformations, according to the admissibility conditions, for a given TSSK sequence, SSSK sequences can be generated by the following theorem.

Theorem 6. Suppose that $(K_C K_E K_D) = (ZEXDYC)$ is an arbitrary TSSK sequence, then four different types of SSSK sequences $(K_C|K_D|K_E) = (YC|XD|ZE)$ can be generated as follows:

$$\text{(I)} \begin{cases} K_C = (\sigma_I^0(ZEXDYC)) *_{2,I^1} C = ZE^{+\tau(Z)}XD^{-\tau(X)}YC \\ K_D = (\sigma_I^1(ZEXDYC)) *_{1,I^2} D = YC^{-\tau(Y)}ZE^{+\tau(Z)}XD \\ K_E = (\sigma_I^2(ZEXDYC)) *_{2,I^0} E = XD^{+\tau(X)}YC^{+\tau(Y)}ZE \end{cases} \tag{5.1}$$

$$\text{(II)} \begin{cases} K_C = (\sigma_I^0(ZEXDYC)) *_{2,I^1} C = ZE^{+\tau(Z)}XD^{-\tau(X)}YC \\ K_D = (\sigma_I^1(ZEXDYC)) *_{2,I^2} D = YC^{-\tau(Y)}ZE^{-\tau(Z)}XD \\ K_E = (\sigma_I^2(ZEXDYC)) *_{1,I^0} E = XD^{+\tau(X)}YC^{-\tau(Y)}ZE \end{cases} \tag{5.2}$$

$$\text{(III)} \begin{cases} K_C = (\sigma_I^0(ZEXDYC)) *_{2,I^1} C = ZE^{+\tau(Z)}XD^{-\tau(X)}YC \\ K_D = (\sigma_I^1(ZEXDYC)) *_{2,I^2} D = YC^{-\tau(Y)}ZE^{-\tau(Z)}XD \\ K_E = (\sigma_I^2(ZEXDYC)) *_{2,I^0} E = XD^{+\tau(X)}YC^{+\tau(Y)}ZE \end{cases} \tag{5.3}$$

$$\text{(IV)} \begin{cases} K_C = (\sigma_I^0(ZEXDYC)) *_{1,I^1} C = ZE^{+\tau(Z)}XD^{+\tau(X)}YC \\ K_D = (\sigma_I^1(ZEXDYC)) *_{2,I^2} D = YC^{-\tau(Y)}ZE^{-\tau(Z)}XD \\ K_E = (\sigma_I^2(ZEXDYC)) *_{2,I^0} E = XD^{+\tau(X)}YC^{+\tau(Y)}ZE. \end{cases} \tag{5.4}$$

The proof of theorem 6 can be completed by the admissibility conditions and lemma 1.

Theorem 7. Suppose that $(K_D K_E K_C) = (ZEYCXD)$ is an arbitrary TSSK sequence, then four types of SSSK sequences $(K_D|K_C|K_E) = (XD|YC|ZE)$ can be generated as follows:

$$\text{(I)} \begin{cases} K_D = (\sigma_{II}^0(ZEYCXD)) *_{1,II^1} D = ZE^{-\tau(Z)}YC^{+\tau(Y)}XD \\ K_C = (\sigma_{II}^1(ZEYCXD)) *_{2,II^2} C = XD^{+\tau(X)}ZE^{+\tau(Z)}YC \\ K_E = (\sigma_{II}^2(ZEYCXD)) *_{2,II^0} E = YC^{+\tau(Y)}XD^{-\tau(X)}ZE \end{cases} \tag{5.5}$$

$$\text{(II)} \begin{cases} K_D = (\sigma_{II}^0(ZEYCXD)) *_{2,II^1} D = ZE^{-\tau(Z)}YC^{-\tau(Y)}XD \\ K_C = (\sigma_{II}^1(ZEYCXD)) *_{1,II^2} C = XD^{+\tau(X)}ZE^{-\tau(Z)}YC \\ K_E = (\sigma_{II}^2(ZEYCXD)) *_{2,II^0} E = YC^{+\tau(Y)}XD^{-\tau(X)}ZE \end{cases} \tag{5.6}$$

$$\text{(III)} \begin{cases} K_D = (\sigma_{II}^0(ZEYCXD)) *_{2,II^1} D = ZE^{-\tau(Z)}YC^{-\tau(Y)}XD \\ K_C = (\sigma_{II}^1(ZEYCXD)) *_{2,II^2} C = XD^{+\tau(X)}ZE^{+\tau(Z)}YC \\ K_E = (\sigma_{II}^2(ZEYCXD)) *_{2,II^0} E = YC^{+\tau(Y)}XD^{-\tau(X)}ZE \end{cases} \tag{5.7}$$

$$\text{(IV)} \begin{cases} K_D = (\sigma_{II}^0(ZEYCXD)) *_{2,II^1} D = ZE^{-\tau(Z)}YC^{-\tau(Y)}XD \\ K_C = (\sigma_{II}^1(ZEYCXD)) *_{2,II^2} C = XD^{+\tau(X)}ZE^{+\tau(Z)}YC \\ K_E = (\sigma_{II}^2(ZEYCXD)) *_{3,II^0} E = YC^{+\tau(Y)}XD^{+\tau(X)}ZE. \end{cases} \tag{5.8}$$

The proof of theorem 7 can be completed by the admissibility conditions and lemma 2.

In theorems 6 and 7, we have four kinds of star transformations to generate SSSK sequences for each of the cyclic ways from TSSK sequences. This enhances the variety of genealogy of star transformations.

The three SSSK sequences in each of the above eight types are compatible. Further, due to the continuity one can obtain the nonsuperstable kneading sequences with star transformations again, therefore the upper and lower sequences of periodic window of SSSK sequences are obtained. For instance, from TSSK sequence $(K_C K_E K_D) = (ZEXDYC)$, the SSSK sequences generated and their non-superstable kneading sequences are as follows:

$$(ZEXDYC) \begin{cases} *_{2,I^1} C = ZE^{+\tau(Z)}XD^{-\tau(X)}YC \\ *_{2,I^0} C = ZE^{-\tau(Z)}XD^{-\tau(X)}YC \end{cases}$$

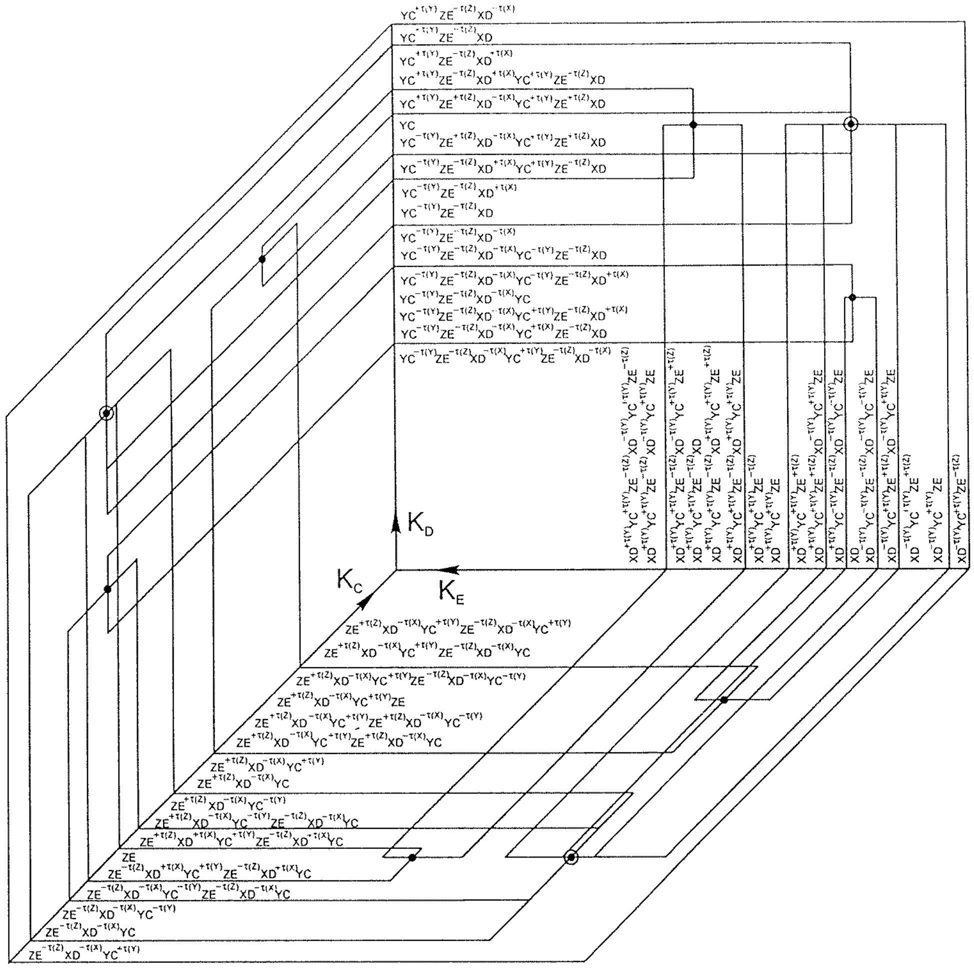

Figure 3. A sketch of three kinds of period-doubling and their cascade for the TSSK sequence $ZEXDYC$. The small circles represent projections of the joint of sequence $ZEXDYC$ to the three kneading coordinate planes. The black circles represent projections of the joints of three period-doubling TSSK sequences of $ZEXDYC$ to the three kneading coordinate planes.

$$ZE^{+\tau(Z)}XD^{-\tau(X)}YC \begin{cases} *_{1,I^1}E = ZE^{+\tau(Z)}XD^{-\tau(X)}YC^{+\tau(Y)} \\ *_{1,I^0}E = ZE^{+\tau(Z)}XD^{-\tau(X)}YC^{-\tau(Y)} \end{cases}$$

$$ZE^{-\tau(Z)}XD^{-\tau(X)}YC \begin{cases} *_{1,I^0}E = ZE^{-\tau(Z)}XD^{-\tau(X)}YC^{-\tau(Y)} \\ *_{1,I^1}E = ZE^{-\tau(Z)}XD^{-\tau(X)}YC^{+\tau(Y)} \end{cases}$$

$$(YCZEXD) \begin{cases} *_{2,I^2}D = YC^{-\tau(Y)}ZE^{-\tau(Z)}XD \\ *_{2,I^1}D = YC^{+\tau(Y)}ZE^{-\tau(Z)}XD \end{cases}$$

$$YC^{-\tau(Y)}ZE^{-\tau(Z)}XD \begin{cases} *_{1,I^2}C = YC^{-\tau(Y)}ZE^{-\tau(Z)}XD^{-\tau(X)} \\ *_{1,I^1}C = YC^{-\tau(Y)}ZE^{-\tau(Z)}XD^{+\tau(X)} \end{cases}$$

$$YC^{+\tau(Y)}ZE^{-\tau(Z)}XD\begin{cases}*_{1,I^1}C = YC^{+\tau(Y)}ZE^{-\tau(Z)}XD^{+\tau(X)}\\ *_{1,I^2}C = YC^{+\tau(Y)}ZE^{-\tau(Z)}XD^{-\tau(X)}\end{cases}$$

$$(XDYCZE)\begin{cases}*_{2,I^0}E = XD^{+\tau(X)}YC^{+\tau(Y)}ZE\\ *_{2,I^2}E = XD^{-\tau(X)}YC^{+\tau(Y)}ZE\end{cases}$$

$$XD^{+\tau(X)}YC^{+\tau(Y)}ZE\begin{cases}*_{1,I^0}D = XD^{+\tau(X)}YC^{+\tau(Y)}ZE^{-\tau(Z)}\\ *_{1,I^2}D = XD^{+\tau(X)}YC^{+\tau(Y)}ZE^{+\tau(Z)}\end{cases}$$

$$XD^{-\tau(X)}YC^{+\tau(Y)}ZE\begin{cases}*_{1,I^2}D = XD^{-\tau(X)}YC^{+\tau(Y)}ZE^{+\tau(Z)}\\ *_{1,I^0}D = XD^{-\tau(X)}YC^{+\tau(Y)}ZE^{-\tau(Z)}.\end{cases}$$

Similarly, we can produce a series of superstable and non-superstable kneading sequences from star transformations for the period-doubling sequences of TSSK sequences and obtain the upper and lower sequences of the windows of the period-doubling sequences. The upper boundaries of windows of the basic periodic sequences $K_C = ZEXDYC$ and $K_E = XDYCZE$ attach with the lower boundaries of windows of their period-doubling sequences $K_C' = ZE^{+\tau(Z)}XD^{-\tau(X)}YC^{+\tau(Y)}ZEXDYC$ and $K_E' = XD^{+\tau(X)}YC^{+\tau(Y)}ZE^{-\tau(Z)}\ XDYCZE$. While the lower boundary of the window of the basic periodic sequence $K_D = YCZEXD$ attaches to the upper boundary of the window of its period-doubling sequence $K_D' = YC^{-\tau(Y)}ZE^{-\tau(Z)}XD^{-\tau(X)}YCZEXD$. Thus the cascade occurs. The period-doubling bifurcations occur at the non-superstable kneading sequences $ZE^{+\tau(Z)}XD^{-\tau(X)}YC^{+\tau(Y)}$, $XD^{+\tau(X)}YC^{+\tau(Y)}ZE^{-\tau(Z)}$ and $YC^{-\tau(Y)}ZE^{-\tau(Z)}XD^{-\tau(X)}$ that are all odd strings. The relation of the period-doubling cascade is shown in figure 3.

In the symbolic dynamics of four letters, the varieties of star transformations of period-tripling and period-doubling of TSSK sequences are obtained. They may help to find the general rule of star products of four letters. However, what we present here is only a clue to solving the problem. We predict that the complete star products in the symbolic dynamics of four letters would be much more complicated than that of the symbolic dynamics of three letters, because of the exponential growth of the kinds of star transformations.

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Appendix. The detailed proof of (I) in lemma 1

We first introduce a term, *common leading string*, that will be used frequently in the proof. For two arbitrary words A, B and $|A| > |B|$, we seek their maximal common part, namely, their intersection H, when the first letters of two words are aligned from the head. If H is not empty it is called the common leading string. If $H = B$, B is called the *leading string of A*.

Assuming $G_1 \in \mathfrak{B}_{L,M}(ZE^{+\tau(Z)})$ in (3.4), from (3.7*a*), if G_1 is not the leading string of ZE, then (3.4) holds. Otherwise G_1 is odd, denoting $ZE^{+\tau(Z)} = G_1Q_1$, (3.4) reduces to

$$XD^{\pm\tau(X)}YC^{\pm\tau(Y)}ZE > Q_1XD^{-\tau(X)}YC^{+\tau(Y)} \tag{A.1}$$

where $Q_1 \in \mathfrak{B}_{N,R}(ZE^{+\tau(Z)})$. Because both Q_1 and $XD^{+\tau(Z)}$ are even, from (3.7*c*) we have that Q_1 is not the leading string of XD, and $XD^{+\tau(Z)}$ not the leading string of Q_1, so (A.1) holds. If $XD^{-\tau(Z)}$ is not the leading string of Q_1, (A.1) also holds. If $XD^{-\tau(Z)}$ is the leading string of Q_1, denoting $Q_1 = XD^{-\tau(Z)}Q_1^{'}$, (A.1) reduces to

$$YC^{\pm\tau(Y)}ZE < Q_1^{'}XD^{-\tau(X)}YC^{+\tau(Y)} \tag{A.2}$$

where $Q_1^{'} \in \mathfrak{B}_{M,N}(ZE^{+\tau(Z)})$. Because $Q_1^{'}$ is odd and $YC^{-\tau(Y)}$ even, from (3.7*b*) we have that $Q_1^{'}$ is not the leading string of YC, and $YC^{-\tau(Y)}$ is not the leading string of $Q_1^{'}$, so (A.2) is valid. If $YC^{+\tau(Y)}$ is not the leading string of $Q_1^{'}$, (A.2) also holds. If $YC^{+\tau(Y)}$ is the leading string of $Q_1^{'}$, denoting $Q_1^{'} = YC^{+\tau(Y)}Q_1^{''}$, (A.2) reduces to

$$ZE > Q_1^{''}XD^{-\tau(X)}YC^{+\tau(Y)} \tag{A.3}$$

where $Q_1^{''} \in \mathfrak{B}_{L,M}(ZE^{+\tau(Z)})$. Because $Q_1^{''}$ is even, from (3.7*a*) we have that $Q_1^{''}$ is not the leading string of ZE, so (A.3) is valid.

Assuming $S_1 \in \mathfrak{B}_{L,M}(XD^{\pm\tau(X)})$ in (3.4). From (3.7*a*) we have that $ZE^{+\tau(Z)}$ is not the leading string of S_1. If S_1 is not the leading string of ZE, then (3.4) holds. Otherwise S_1 is even, denoting $ZE^{+\tau(Z)} = S_1T_1$, and (3.4) reduces to

$$YC^{\pm\tau(Y)}ZE < T_1XD^{-\tau(X)}YC^{+\tau(Y)} \tag{A.4}$$

where $T_1 \in \mathfrak{B}_{M,N}(ZE^{+\tau(Z)})$. Because T_1 is odd and $YC^{-\tau(Y)}$ even, from (3.7*b*) we have that T_1 is not the leading string of YC and $YC^{-\tau(Y)}$ is not the leading string of T_1, so (A.4) holds. If $YC^{+\tau(Y)}$ is not the leading string of T_1, (A.4) also holds. If $YC^{+\tau(Y)}$ is the leading string of T_1, denoting $T_1 = YC^{+\tau(Y)}T_1^{'}$, (A.4) reduces to

$$ZE > T_1^{'}XD^{-\tau(X)}YC^{+\tau(Y)} \tag{A.5}$$

where $T_1^{'} \in \mathfrak{B}_{L,M}(ZE^{+\tau(Z)})$. Because $T_1^{'}$ is even, from (3.7*a*) we have that $T_1^{'}$ is not the leading string of ZE and (A.5) is valid.

Assuming $U_1 \in \mathfrak{B}_{L,M}(YC^{\pm\tau(Y)})$ in (3.4), we have from (3.7*a*) that $ZE^{+\tau(Z)}$ is not the leading string of U_1. If U_1 is not the leading string of ZE then (3.4) holds. Otherwise U_1 is odd, denoting $ZE^{+\tau(Z)} = U_1V_1$ and (3.4) reduces to

$$ZE > V_1XD^{-\tau(X)}YC^{+\tau(Y)} \tag{A.6}$$

where $V_1 \in \mathfrak{B}_{L,M}(ZE^{+\tau(Z)})$. Because V_1 is even, from (3.7*a*) we have that V_1 is not the leading string of ZE, $(A.6)$ is valid.

The proof of (3.4) is completed.

Assuming $G_2 \in \mathfrak{B}_{L,M}(YC^{-\tau(Y)})$ in (3.5*a*), we have from (3.7*a*) that $ZE^{+\tau(Z)}$ is not the leading string of G_2. If G_2 is not the leading string of ZE, then (3.5*a*) holds. Otherwise G_2 is odd, denoting $ZE^{+\tau(Z)} = G_2Q_2$, (3.5*a*) reduces to

$$ZE^{+\tau(Z)}XD^{\pm\tau(X)}YC > Q_2XD^{-\tau(X)}YC^{+\tau(Y)} \tag{A.7}$$

where $Q_2 \in \mathfrak{B}_{L,M}(ZE^{+\tau(Z)})$. Because Q_2 is even, from (3.7*a*) we have that Q_2 is not the leading string of ZE and (A.7) is valid.

Assuming $S_2 \in \mathfrak{B}_{L,M}(ZE^{+\tau(Z)})$ in (3.5*a*). From (3.7*a*), if S_2 is not the leading string of ZE, then (3.5*a*) holds. Otherwise S_2 is odd, denoting $ZE^{+\tau(Z)} = S_2T_2$ and (3.5*a*) reduces to

$$XD^{\pm\tau(X)}YC > T_2XD^{-\tau(X)}YC^{+\tau(Y)} \tag{A.8}$$

where $T_2 \in \mathfrak{B}_{N,R}(ZE^{+\tau(Z)})$. Because T_2 and $XD^{+\tau(X)}$ are even, from (3.7c) we have that T_2 is not the leading string of XD and $XD^{+\tau(X)}$ is not the leading string of T_2, so (A.8) is valid. If $XD^{-\tau(X)}$ is not the leading string of T_2, (A.8) also holds. If $XD^{-\tau(X)}$ is the leading string of T_2, denoting $T_2 = XD^{-\tau(X)}T_2^{'}$, (A.8) reduces to

$$YC < T_2^{'}XD^{-\tau(X)}YC^{+\tau(Y)} \tag{A.9}$$

where $T_2^{'} \in \mathfrak{B}_{M,N}(ZE^{+\tau(Z)})$. Because $T_2^{'}$ is odd, we have from (3.7b) that $T_2^{'}$ is not the leading string of YC and (A.9) is valid.

Assuming $U_2 \in \mathfrak{B}_{L,M}(XD^{\pm\tau(X)})$ in (3.5a) we have from (3.7a) that $ZE^{+\tau(Z)}$ is not the leading string of U_2. If U_2 is not the leading string of ZE then (3.5a) holds. Otherwise U_2 is even, denoting $ZE^{+\tau(Z)} = U_2V_2$ and (3.5a) reduces to

$$YC < V_2XD^{-\tau(X)}YC^{+\tau(Y)} \tag{A.10}$$

where $V_2 \in \mathfrak{B}_{M,N}(ZE^{+\tau(Z)})$. Because V_2 is odd, we have from (3.7b) that V_2 is not the leading string of YC and (A.10) is valid.

The proof of (3.5a) is completed.

Assuming $G_3 \in \mathfrak{B}_{L,M}(YC^{-\tau(Y)})$ in (3.5b), from (3.7a), if G_3 is not the leading string of ZE, then (3.5b) holds. Otherwise G_3 is odd, denoting $ZE = G_3Q_3E$, (3.5b) reduces to

$$ZE^{-\tau(Z)}XD^{\pm\tau(X)}YC > Q_3E \tag{A.11}$$

where $Q_3E \in \mathfrak{B}_{L,M}(ZE)$. From (3.7$a$) we have (A.11).

Assuming $S_3 \in \mathfrak{B}_{L,M}(ZE^{-\tau(Z)})$ in (3.5b), from (3.7a), if S_3 is not the leading string of ZE, then (3.5b) holds. Otherwise S_3 is odd. Denoting $ZE = S_3T_3E$, (3.5b) reduces to

$$XD^{\pm\tau(X)}YC > T_3E \tag{A.12}$$

where $T_3E \in \mathfrak{B}_{N,R}(ZE)$. From (3.7$c$) we have that $XD^{+\tau(X)}$ is not the leading string of T_3E. If $XD^{-\tau(X)}$ is not the leading string of T_3E, then (A.12) is valid. Otherwise, denoting $T_3E = XD^{-\tau(X)}T_3^{'}E$, (A.12) reduces to

$$YC < T_3^{'}E \tag{A.13}$$

where $T_3^{'}E \in \mathfrak{B}_{M,N}(ZE)$. From (3.7$b$) we have (A.13).

Assuming $U_3 \in \mathfrak{B}_{L,M}(XD^{\pm\tau(X)})$ in (3.5b), from (3.7a), if U_3 is not the leading string of ZE, then (3.5b) holds. Otherwise U_3 is even, denoting $ZE = U_3V_3E$, (3.5b) reduces to

$$YC < V_3E \tag{A.14}$$

where $V_3E \in \mathfrak{B}_{M,N}(ZE)$. From (3.7$b$) we have (A.14).

The proof of (3.5b) is completed.

Assuming $G_4 \in \mathfrak{B}_{L,M}(XD^{+\tau(X)})$ in (3.6a), from (3.7a), if G_4 is not the leading string of ZE, then (3.6a) holds. Otherwise G_4 is even. Denoting $ZE^{+\tau(Z)} = G_4Q_4$, (3.6a) reduces to

$$YC^{\pm\tau(Y)}ZE^{+\tau(Z)}XD < Q_4XD^{-\tau(X)}YC^{+\tau(Y)} \tag{A.15}$$

where $Q_4 \in \mathfrak{B}_{M,N}(ZE^{+\tau(Z)})$. Because Q_4 is odd and $YC^{-\tau(Y)}$ even, from (3.7b) we have that Q_4 is not the leading string of YC and $YC^{-\tau(Y)}$ is not the leading string of Q_4, so (A.15) is valid. If $YC^{+\tau(Y)}$ is not the leading string of Q_4, (A.15) also holds. If $YC^{+\tau(Y)}$ is the leading string of Q_4, denoting $Q_4 = YC^{+\tau(Y)}Q_4^{'}$, (A.15) reduces to

$$ZE^{+\tau(Z)}XD > Q_4^{'}XD^{-\tau(X)}YC^{+\tau(Y)} \tag{A.16}$$

where $Q_4^{'} \in \mathfrak{B}_{L,M}(ZE^{+\tau(Z)})$. Because $Q_4^{'}$ is even, we have from (3.7a) that $Q_4^{'}$ is not the leading string of ZE and (A.16) is valid.

Assuming $S_4 \in \mathfrak{B}_{L,M}(YC^{\pm\tau(Y)})$ in (3.6*a*), from (3.7*a*), if S_4 is not the leading string of ZE, then (3.6*a*) holds. Otherwise S_4 is odd. Denoting $ZE^{+\tau(Z)} = S_4T_4$, (3.6*a*) reduces to

$$ZE^{+\tau(Z)}XD > T_4XD^{-\tau(X)}YC^{+\tau(Y)} \tag{A.17}$$

where $T_4 \in \mathfrak{B}_{L,M}(ZE^{+\tau(Z)})$. Because T_4 is even, from (3.7*a*) we have that T_4 is not the leading string of ZE and (A.17) is valid.

Assuming $U_4 \in \mathfrak{B}_{L,M}(ZE^{+\tau(Z)})$ in (3.6*a*), from (3.7*a*), if U_4 is not the leading string of ZE, then (3.6*a*) holds. Otherwise U_4 is odd. Denoting $ZE^{+\tau(Z)} = U_4V_4$, (3.6*a*) reduces to

$$XD > V_4XD^{-\tau(X)}YC^{+\tau(Y)} \tag{A.18}$$

where $V_4 \in \mathfrak{B}_{N,R}(ZE^{+\tau(Z)})$. Because V_4 is even, from (3.7*c*) we have that V_4 is not the leading string of XD and (A.18) is valid.

The proof of (3.6*a*) is completed.

Assuming $G_5 \in \mathfrak{B}_{L,M}(XD^{+\tau(X)})$ in (3.6*b*), from (3.7*a*), if G_5 is not the leading string of ZE, then (3.6*b*) holds. Otherwise G_5 is even. Denoting $ZE = G_5Q_5E$, (3.6*b*) reduces to

$$YC^{\pm\tau(Y)}ZE^{-\tau(Z)}XD < Q_5E \tag{A.19}$$

where $Q_5E \in \mathfrak{B}_{M,N}(ZE)$. From (3.7*b*) we have that $YC^{-\tau(Y)}$ is not the leading string of Q_5E. If $YC^{+\tau(Y)}$ is not the leading string of Q_5E, then (A.19) holds. If $YC^{+\tau(Y)}$ is the leading string of Q_5E, denoting $Q_5E = YC^{+\tau(Y)}Q_5'E$, (A.19) reduces to

$$ZE^{-\tau(Z)}XD > Q_5'E \tag{A.20}$$

where $Q_5'E \in \mathfrak{B}_{L,M}(ZE)$. (A.20) is valid from (3.7*a*).

Assuming $S_5 \in \mathfrak{B}_{L,M}(YC^{\pm\tau(Y)})$ in (3.6*b*), from (3.7*a*), if S_5 is not the leading string of ZE, then (3.6*b*) holds. Otherwise S_5 is odd. Denoting $ZE = S_5T_5E$, (3.6*b*) reduces to

$$ZE^{-\tau(Z)}XD > T_5E \tag{A.21}$$

where $T_5E \in \mathfrak{B}_{L,M}(ZE)$. (A.21) holds from (3.7*a*).

Assuming $U_5 \in \mathfrak{B}_{L,M}(ZE^{-\tau(Z)})$ in (3.6*b*), from (3.7*a*), if U_5 is not the leading string of ZE, then (3.6*b*) holds. Otherwise U_5 is odd. Denoting $ZE = U_5V_5E$, (3.6*b*) reduces to

$$XD > V_5E \tag{A.22}$$

where $V_5E \in \mathfrak{B}_{N,R}(ZE)$. (A.22) holds from (3.7*c*).

The proof of (3.6*b*) is completed.

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